Calculus of Variations in L^∞

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Outline

Introduction

Calculus of variations: find minimisers of functionals $\mathcal{F}\colon X \to \mathbb{R}$ where X is some (carefully chosen) function space

Examples:

- Brachistochrone problem
- Elastica problem
- Isoperimetric problem
- Willmore conjecture / Helfrich energy
- Optimal transport
- Various physics/applied problems

Normally $\mathcal{F}$ is an integral:

$$\begin{aligned} \text{Elastica problem}: &\quad \int_\gamma \kappa^2 \, \mathrm{d}s \\ \text{Optimal transport}: &\quad \int_{X \times Y} c(x,y) \, \mathrm{d}\gamma(x,y) \\ \text{Willmore conjecture}: &\quad \int_\Sigma H^2 \, \mathrm{d}\mu \end{aligned}$$

but this does not have to be the case...

Calculus of Variations in L^∞ ("L^∞ CoV") is the study of variational problems where the functional $\mathcal{F}$ is a supremum (i.e. L^∞ norm):

$$\infty\text{-elastica problem: } \operatorname*{ess\,sup}_{\gamma} |\kappa| = \|\kappa\|_{L^\infty},$$

$$L^\infty \text{ Optimal transport: } \gamma - \operatorname*{ess\,sup}_{(x,y)} |c(x,y)| = \|c\|_{L^\infty}$$

$$\infty\text{-Willmore surfaces: } \operatorname*{ess\,sup}_{\Sigma} |H| = \|H\|_{L^\infty}$$

Important Milestones:
1960's: Aronsson: simple, first-order functionals
2010's: second order functionals, geometric analysis

Motivation:
Fundamental interest; generalisation of interesting 'standard' CoV
Use in applications– "minimises maximum error"

Euler-Lagrange Equations

How do we analyse solutions of variational problems?
First: solutions have to exist!

“Direct method”: take minimising sequence, show it converges to a minimiser

This is technical so ignore it here

Euler-Lagrange equations:

Critical points x of $f\colon \mathbb{R} \to \mathbb{R}$ solve $\frac{\mathrm{d}}{\mathrm{d}\varepsilon} f(x+\varepsilon)\big|_{\varepsilon=0} = 0$
Critical points x of $f\colon \mathbb{R}^n \to \mathbb{R}$ solve $\frac{\mathrm{d}}{\mathrm{d}\varepsilon} f(x+\varepsilon v)\big|_{\varepsilon=0} = 0$ for all $v \in \mathbb{R}^n$
So critical points f of $\mathcal{F}$ should also have "zero derivative in every direction" i.e.

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{F}[f+\varepsilon\phi]\Big|_{\varepsilon=0} = 0$$

for all "suitable" functions ϕ.

Example: shortest path between two points

L^p Approximation

Problem: for L^∞ problems, $\mathcal{F}$ isn't differentiable! So

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{F}[f+\varepsilon\phi]\Big|_{\varepsilon=0}=0$$

is nonsense.

Solution: "L^p approximation"

Recap: L^p spaces:

L^p norm: weighted average of f over a domain Ω:

$$\|f\|_{L^p} = \left(\int_\Omega |f|^p \,\mathrm{d}x \right)^{\frac{1}{p}}$$

and $L^p(\Omega) := \{ f \,:\, \|f\|_{L^p} < \infty \}$.

In the limit, all the weight goes to the extreme points of f:

$$\lim_{p \to \infty} \|f\|_{L^p} = \|f\|_{L^\infty} = \operatorname{ess\,sup} |f|$$

Idea: $\|\cdot\|_{L^p}$ "approximates" $\|\cdot\|_{L^\infty}$.
So we approximate the non-differentiable $\mathcal{F} = \|\cdot\|_{L^\infty}$ by the differentiable $\|\cdot\|_{L^p}$.
This is the idea of L^p approximation:

1. Consider L^p problem for $p \in [1, \infty)$
2. Compute Euler-Lagrange for L^p problem
3. Send $p \to \infty$, get convergence of

$$\begin{cases} L^p \text{ minimisers} & \to \quad L^\infty \text{ minimiser} \\ L^p \text{ E-L equations} & \to \quad \text{``}L^\infty \text{ E-L'' equation} \quad \text{(hard)} \end{cases}$$

4. Analyse limiting equation to learn about L^∞ minimisers

L^p approximation originated in the 60's with Aronsson and is a well-known tool in L^∞ CoV

I use it for my research (finding shapes which minimise L^∞ curvature)

Example Problem

Minimise the quantity $\|f''\|_{L^\infty}$ over all functions $f\colon [0,1] \to \mathbb{R}$ with given boundary data up to first derivatives:

$$f(0) = a_1, f(1) = a_2, f'(0) = b_1, f'(1) = b_2.$$

Technically, we minimise over the function space $W^{2,\infty}_{f_0}(0,1)$.

(Assume no 'trivial solutions' exist, i.e. no straight lines)

Approximating problem:
Minimise the quantity $\|f''\|_{L^p}$ over $W^{2,p}_{f_0}(0,1)$.

Steps 1 & 2: approximating problem:
Minimise the quantity $\|f''\|_{L^p}$ over $W^{2,p}_{f_0}(0,1)$.

Computations show that a minimiser f_p satisfies the Euler-Lagrange equation

$$\left(|f_p''|^{p-2} f_p''\right)'' = 0.$$

Step 3:

We know $\left(|f_p''|^{p-2}f_p''\right)'' = 0$, but $|f_p''|^{p-2}f_p''$ may behave badly as $p \to \infty$ (either blow up to ∞ or shrink to 0). We normalise to stop this happening:

Set $g_p = C_p|f_p''|^{p-2}f_p''$ with the normalisation constant C_p such that the system

$$g_p'' = 0,$$

$$f_p'' = \|f_p''\|_{L^p}|g_p|^{\frac{1}{p-1}}\frac{g_p}{|g_p|}$$

holds.

After some analysis, we get the convergence we need: (f_p), (g_p) converge to functions f and g such that:

- f is a minimiser of our L^∞ problem,
- The system of equations

$$g'' = 0,$$
$$|g|f'' = \|f''\|_{L^\infty} g$$

is satisfied.

Step 4:

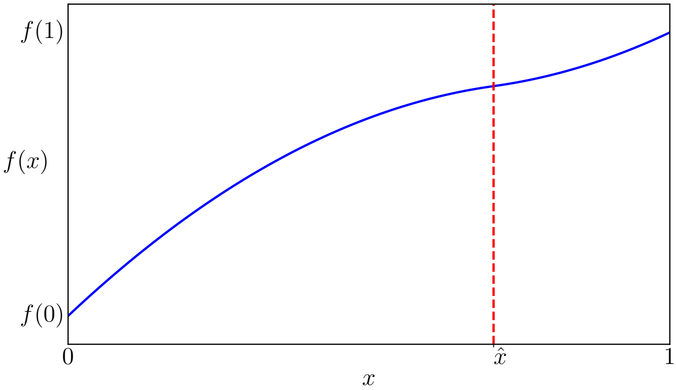
Since $g'' = 0$, g must be linear i.e. $g(x) = ax + b$.
From $|g|f'' = \|f''\|_{L^\infty} g$ we find:

- Where $g > 0$, $f'' = +\|f''\|_{L^\infty}$;
- Where $g < 0$, $f'' = -\|f''\|_{L^\infty}$;
- $g = 0$ happens at a single point so it doesn't matter.

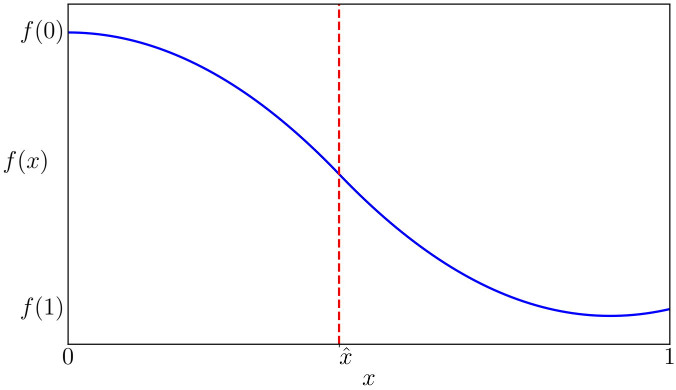
We conclude that f is made up of two parabolas, 'joined together' at the zero of g in such a way that the values of f and f' match

After some (messy) maths, we can actually get an explicit expression for f in terms of the boundary data.

Example where $f(0) = 0$, $f(1) = 1$, $f'(0) = 2$, $f'(1) = 1$:



Example where $f(0) = 1$, $f(1) = -1$, $f'(0) = 0$, $f'(1) = 1$:



This example is simple but highlights some characteristic features of L^∞ minimisers:

- Low regularity
- Magnitude either constant or zero
- Sign governed by limiting equations

My Work

Interface of geometric analysis & L^∞ CoV: minimising L^∞ curvature
e.g. among all curves with fixed length and boundary data, which ones minimise L^∞ curvature? What do they look like?
2D problem: minimise L^∞ norm of *mean curvature* of surface
Interesting topological/analytical problems